

# Mass-Deformed BLG Theory in Light-Cone Superspace

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## Abstract

Maximally supersymmetric mass deformation of the Bagger-Lambert-Gustavsson (BLG) theory corresponds to a *non-central* extension of the  $d = 3$   $N = 8$  Poincaré superalgebra (allowed in three dimensions). We obtain its light-cone superspace formulation which has a novel feature of the dynamical supersymmetry generators being *cubic* in the kinematical ones. The mass deformation picks a quaternionic direction, described by  $\Omega_m^n$ , which breaks the  $SO(8)$   $R$ -symmetry down to  $SO(4) \times SO(4)$ . The Hamiltonian of the theory is shown to be a quadratic form of the dynamical supersymmetry transformations, to all orders in the mass parameter,  $M$ , and the structure constants,  $f^{abcd}$ .

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## 1 Introduction

The Bagger-Lambert-Gustavsson (BLG) theory [1, 2, 3] is the maximally supersymmetric and superconformal three-dimensional gauge theory of Chern-Simons type. It has rigid symmetry described by the superconformal group  $OSp(2, 2|8)$  and local symmetry described by the 3-Lie algebra with structure constants  $f^{bcd}{}_a$ . This theory has a very interesting mass deformation [4, 5], which breaks the conformal symmetries while preserving all of the maximal supersymmetry. The resulting symmetry group is a *non-central* extension of the  $d = 3$   $N = 8$  superPoincaré group [6, 7]. Such an extension is forbidden in four and higher dimensions [8, 9], which makes this three-dimensional theory very special.

As is well-known, supersymmetric theories enjoy living in superspace. The latter comes in two varieties: off-shell superspace [10] (best suited for minimal supersymmetry) and on-shell superspace [11] (best suited for maximal supersymmetry [12]). The former requires additional auxiliary fields, whereas the latter operates with only physical degrees of freedom. Light-cone

(LC) superspace is the best known example of on-shell superspace. It has been used, in particular, to prove the UV finiteness of the  $N = 4$  super-Yang-Mills [13, 14]. One surprising feature of the LC superspace is that the LC superspace Hamiltonian of a maximally supersymmetric theory (in all the cases studied to date) appears to be a quadratic form of the dynamical supersymmetry transformation of the basic superfield [15, 16, 17, 18]. In the LC superspace formulation of the BLG theory [19, 17, 18], this has been verified [18] to linear order in  $f^{abcd}$ . In this paper, we will prove that this property holds to all orders in  $f^{abcd}$ .

Attempting to construct the LC superspace formulation of the mass-deformed BLG theory from scratch, by solving the constraints imposed by the symmetry group (as in [18]), one would encounter (at least) two problems. First, the dynamical supersymmetry generators,  $\mathcal{Q}$ 's, must be *cubic* in the kinematical supersymmetry generators,  $q$ 's, because in the commutator of two supersymmetries one must find an  $R$ -symmetry transformation which is quadratic in  $q$ 's. At the same time, in the commutator of two  $\mathcal{Q}$ 's there should be no terms quartic in  $q$ 's, as no such symmetry generators exist. The apparent quartic terms must somehow cancel. Second, the mass deformation should break the  $SO(8)$   $R$ -symmetry of the BLG theory down to  $SO(4) \times SO(4)$ . And it is not a priori obvious how to accomplish this in the LC superspace setting best suited for keeping  $SU(4)$   $R$ -symmetry manifest.

In this paper, we will perform the top-down reduction [17] of the known (covariantly formulated) mass-deformed BLG theory [4, 5] to its LC superspace form. This allows us to solve the above mentioned problems. In the covariant formulation, the mass parameter is accompanied by the  $32 \times 32$  matrix  $\Gamma_{3456}$  (the product of four 11-dimensional gamma matrices) which in our conventions is

$$\Gamma_{3456} = i \begin{pmatrix} + & & & \\ & - & & \\ & & + & \\ & & & - \end{pmatrix} \otimes \begin{pmatrix} \Omega_n^m & \\ 0 & \Omega_m^n \end{pmatrix}, \quad (1.1)$$

where  $\Omega_n^m = -\Omega_n^m$  is one of the quaternionic matrices in the algebra of  $SO(4) \subset SU(4)$ . We will see that this matrix intertwines the  $SO(8)$   $R$ -symmetry generators [18] ( $T_m^n, T_{mn}, T^{mn}$  and  $T$ ) in a way that reduces the  $R$ -symmetry group to  $SO(4) \times SO(4)$ . We will also see that  $\mathcal{Q}$ 's are indeed cubic in  $q$ 's, whereas the following identity

$$\delta_{[n}^m \Omega_k^l \Omega_r^s q_t q_s] = 0 \quad (1.2)$$

is responsible for the absence of terms quartic in  $q$ 's inside the commutator of two  $\mathcal{Q}$ 's.

We will see that the mass deformation affects the dynamical supersymmetry transformations, but not the kinematical ones. (Therefore, mass is treated as an interaction in LC superspace.) The modification is fairly simple, given that the mass parameter,  $M$ , never multiplies the structure constants,  $f^{bcd}_a$ . The modification at the Lagrangian level is more involved, but once we show that the LC superspace Hamiltonian of the mass-deformed BLG theory is a quadratic form of the dynamical supersymmetry transformations, the overall structure becomes quite simple.

In Section 2, we analyze the mass-deformed BLG theory with  $f^{bcd}_a$  set to zero (i.e. the “abelian” version of the theory) and with the gauge indices on the fields accordingly suppressed. The indices will be reintroduced together with  $f^{bcd}_a$  in Section 3. Some technical details (includ-

ing the major part of the proof of the quadratic form property of the LC superspace Hamiltonian) are delegated to the Appendices. Throughout the paper, we follow the conventions of [17].

## 2 The abelian theory

In this section, we will analyze the mass-deformed BLG theory [4, 5] with  $f^{bcd}_a = 0$ . In the covariant formulation, the field content of the theory consists of eight scalars,  $X^I$ ,  $I = 3, \dots, 10$ , and a single 32-component Majorana spinor,  $\Psi$ , satisfying an additional constraint  $\Gamma_{012}\Psi = -\Psi$ . On-shell, there are 8 bosonic and 8 fermionic degrees of freedom. They fit nicely into the LC superfield  $\phi$  [12]. Our goal is to find how  $\phi$  varies under supersymmetry transformations. In the covariant formulation, the latter are described by a 32-component Majorana spinor,  $\epsilon$ , satisfying  $\Gamma_{012}\epsilon = +\epsilon$ . In the LC superspace formulation,  $\epsilon$  is reduced to 8 kinematical supersymmetry parameters,  $\alpha^m$  and  $\alpha_m = (\alpha^m)^*$ , and 8 dynamical supersymmetry parameters,  $\beta^m$  and  $\beta_m = (\beta^m)^*$ ;  $m = 1, 2, 3, 4$ . We will see that the mass deformation affects only the latter, and that the mass parameter,  $M$ , appears multiplied by a matrix  $\Omega_m^n$  that breaks the  $SO(8)$   $R$ -symmetry down to  $SO(4) \times SO(4)$ . We will close this section with the discussion of the Hamiltonian,  $H$ , and the dynamical Lorentz boost generator,  $\mathcal{J}^-$ .

### 2.1 Covariant formulation

The action of the abelian mass-deformed BLG theory is  $S = \int d^3x \mathcal{L}$  with

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu X^I)(\partial^\mu X^I) + \frac{i}{2}\bar{\Psi}\Gamma^\mu\partial_\mu\Psi - \frac{1}{2}M^2 X^I X^I + \frac{i}{2}M\bar{\Psi}\Gamma_{3456}\Psi, \quad (2.1)$$

where  $\mu = 0, 1, 2$  and  $I = 3, 4, 5, 6, 7, 8, 9, 10$ . The symmetries of this action are

- translations, with parameters  $v^\mu$ ,

$$\delta_v X^I = v^\mu \partial_\mu X^I, \quad \delta_v \Psi = v^\mu \partial_\mu \Psi; \quad (2.2)$$

- Lorentz transformations, with parameters  $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ ,

$$\delta_\lambda X^I = \lambda^{\mu\nu} x_\mu \partial_\nu X^I, \quad \delta_\lambda \Psi = \lambda^{\mu\nu} x_\mu \partial_\nu \Psi + \frac{1}{4}\lambda^{\mu\nu}\Gamma_{\mu\nu}\Psi; \quad (2.3)$$

- $SO(4) \times SO(4)$   $R$ -symmetry transformations,

$$\delta_R X^i = R^{ij} X^j, \quad \delta_R X^{i'} = R^{i'j'} X^{j'}, \quad \delta_R \Psi = \frac{1}{4}R^{ij}\Gamma_{ij}\Psi + \frac{1}{4}R^{i'j'}\Gamma_{i'j'}\Psi, \quad (2.4)$$

where  $i = 3, 4, 5, 6$  and  $i' = 7, 8, 9, 10$ ; the parameters  $R^{ij} = -R^{ji}$  parametrize the first  $SO(4)$ , and  $R^{i'j'} = -R^{j'i'}$  parametrize the second  $SO(4)$ ;

- supersymmetry transformations, with the parameter  $\epsilon$ ,

$$\delta_\epsilon X^I = i\bar{\epsilon}\Gamma^I\Psi, \quad \delta_\epsilon \Psi = \Gamma^\mu\Gamma^I\epsilon\partial_\mu X^I - M\Gamma_{3456}\Gamma^I\epsilon X^I. \quad (2.5)$$

The algebra of these symmetries closes on-shell, i.e. provided the equations of motion implied by (2.1),

$$(\partial^\mu \partial_\mu - M^2)X^I = 0, \quad (\Gamma^\mu \partial_\mu + M\Gamma_{3456})\Psi = 0, \quad (2.6)$$

are satisfied. The key commutator is that of two supersymmetries, for which we find

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_v + \delta_R, \quad (2.7)$$

where (note that our conventions are such that  $\Gamma_{789(10)} = -\Gamma_{3456}\Gamma_{012}$ )

$$v^\mu = -2i(\bar{\epsilon}_2 \Gamma^\mu \epsilon_1), \quad R^{ij} = 2iM(\bar{\epsilon}_2 \Gamma^{ij} \Gamma_{3456} \epsilon_1), \quad R^{i'j'} = 2iM(\bar{\epsilon}_2 \Gamma^{i'j'} \Gamma_{789(10)} \epsilon_1). \quad (2.8)$$

As the commutator of two supersymmetry transformations yields the  $R$ -symmetry transformation (in addition to the standard translation), this algebra is a *non-central* extension of the  $d = 3$   $N = 8$  superPoincaré algebra.<sup>1</sup>

## 2.2 LC supersymmetry transformations: component form

Using the LC projectors,  $P_+ = -\frac{1}{2}\Gamma_+\Gamma_-$  and  $P_- = -\frac{1}{2}\Gamma_-\Gamma_+$ , we define  $\epsilon_\pm = P_\pm \epsilon$  and  $\Psi_\pm = P_\pm \Psi$  [17]. The fermionic equation of motion in (2.6) can then be used to solve for  $\Psi_-$ ,

$$\Psi_- = \frac{1}{2\partial^+} \Gamma_- (\Gamma_2 \partial + M\Gamma_{3456}) \Psi_+, \quad (2.9)$$

where  $\partial^+ = \frac{1}{\sqrt{2}}(-\partial_0 + \partial_1)$  and  $\partial = \partial_2$ . From now on, only variations of  $\Psi_+$  need to be considered. The transformations (2.5) split into the kinematical supersymmetry transformations,

$$\begin{aligned} \delta_{\epsilon_-} X^I &= i\bar{\epsilon}_- \Gamma^I \Psi_+ \\ \delta_{\epsilon_-} \Psi_+ &= \Gamma_+ \Gamma^I \epsilon_- \partial^+ X^I, \end{aligned} \quad (2.10)$$

and the dynamical supersymmetry transformations,

$$\begin{aligned} \delta_{\epsilon_+} X^I &= \frac{i}{2\partial^+} \bar{\epsilon}_+ \Gamma^I \Gamma_- (\Gamma_2 \partial + M\Gamma_{3456}) \Psi_+ \\ \delta_{\epsilon_+} \Psi_+ &= (\Gamma_2 \partial - M\Gamma_{3456}) \Gamma^I \epsilon_+ X^I. \end{aligned} \quad (2.11)$$

As the mass deformation does not affect the kinematical supersymmetry transformations (2.10), the fitting of the degrees of freedom into the LC superfield  $\phi$  is as in [17]. The eight scalars  $X^I$  define the bosonic components of  $\phi$  as follows,

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}(X^3 + iX^4) \\ C_{mn} &= \frac{1}{\sqrt{2}}(\eta_1 X^5 + \eta_2 X^6 + \eta_3 X^7) + \frac{i}{\sqrt{2}}(\tilde{\eta}_1 X^8 + \tilde{\eta}_2 X^9 + \tilde{\eta}_3 X^{10}), \end{aligned} \quad (2.12)$$

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<sup>1</sup> This superalgebra (with 16 supercharges) is described in Appendix E.2 of [7]. It is a doubled version of the superalgebra with 8 supercharges given explicitly in Appendix E.1 of [7] and described after eq. (48) in [6].

where  $(\eta_{\mathbf{a}})_{mn}$  and  $(\tilde{\eta}_{\mathbf{a}})_{mn}$ ,  $\mathbf{a} = 1, 2, 3$ , are six  $4 \times 4$  matrices (the so-called 't Hooft symbols [20, 21]) whose explicit form and properties are given in Appendix A. The 8 fermionic components in  $\Psi_+$  define the fermionic components of  $\phi$  as follows,

$$\Psi_+ = \begin{pmatrix} \psi_+^m \\ \psi_{m+} \end{pmatrix}, \quad \psi_+^m = \begin{pmatrix} 0 \\ 0 \\ \chi^m \\ 0 \end{pmatrix}, \quad \psi_{m+} = \begin{pmatrix} 0 \\ \chi_m \\ 0 \\ 0 \end{pmatrix}; \quad \chi_m = (\chi^m)^*. \quad (2.13)$$

The 8 fermionic parameters in  $\epsilon_-$  define the kinematical supersymmetry parameters  $\alpha$ ,

$$\epsilon_- = \begin{pmatrix} \epsilon_-^m \\ \epsilon_{m-} \end{pmatrix}, \quad \epsilon_-^m = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha^m \end{pmatrix}, \quad \epsilon_{m-} = \begin{pmatrix} -\alpha_m \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad \alpha_m = (\alpha^m)^*, \quad (2.14)$$

whereas the 8 parameters in  $\epsilon_+$  define the dynamical supersymmetry parameters  $\beta$ ,<sup>2</sup>

$$\epsilon_+ = \begin{pmatrix} \epsilon_+^m \\ \epsilon_{m+} \end{pmatrix}, \quad \epsilon_+^m = \begin{pmatrix} 0 \\ \beta^m \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_{m+} = \begin{pmatrix} 0 \\ 0 \\ \beta_m \\ 0 \end{pmatrix}; \quad \beta_m = (\beta^m)^*. \quad (2.15)$$

The component form of the kinematical supersymmetry transformations is as in [17],

$$\begin{aligned} \delta_{\epsilon_-} A &= i\sqrt{2}\alpha^m \chi_m \\ \delta_{\epsilon_-} \chi_m &= 2\partial^+ (A\alpha_m + C_{mn}\alpha^n) \\ \delta_{\epsilon_-} C_{mn} &= -i\sqrt{2}(\alpha_m \chi_n - \alpha_n \chi_m + \varepsilon_{mnkl}\alpha^k \chi^l), \end{aligned} \quad (2.16)$$

whereas in the dynamical supersymmetry transformations we find  $O(M)$  modifications,

$$\begin{aligned} \delta_{\epsilon_+} A &= -\frac{1}{\partial^+} (\beta^m \partial \chi_m + iM\beta^m \Omega_m{}^n \chi_n) \\ \delta_{\epsilon_+} \chi_m &= -i\sqrt{2}\partial (A\beta_m - C_{mn}\beta^n) - \sqrt{2}M\Omega_m{}^n (A\beta_n - C_{nk}\beta^k) \\ \delta_{\epsilon_+} C_{mn} &= \frac{\partial}{\partial^+} (-\beta_m \chi_n + \beta_n \chi_m + \varepsilon_{mnkl}\beta^k \chi^l) \\ &\quad - \frac{iM}{\partial^+} ((\beta_m \Omega_n{}^k - \beta_n \Omega_m{}^k) \chi_k - \varepsilon_{mnkl}\beta^k \Omega_s{}^l \chi^s), \end{aligned} \quad (2.17)$$

where we defined  $\Omega_m{}^n \equiv (\eta_3)_{mn}$  (see Appendix A), so that<sup>3</sup>

$$\Omega_m{}^n = \begin{pmatrix} & + & \\ - & & \\ & & + \\ & - & \end{pmatrix} = (\Omega_m{}^n)^* \equiv \Omega^m{}_n = -\Omega_n{}^m, \quad \Omega_m{}^k \Omega_k{}^n = -\delta_m^n. \quad (2.18)$$

<sup>2</sup> In [17], the parameters of the dynamical supersymmetry were called  $\eta^m$ . Now we use  $\beta^m$  instead, in order to avoid confusion with the 't Hooft symbols, which are used extensively in this paper.

<sup>3</sup> In the matrix-to-second-rank-tensor correspondence, we assign the left index to the rows and the right index to the columns. Then, for example,  $\Omega_1{}^2 = +1$  and  $\Omega_2{}^1 = -1$ . Also,  $\Omega_m{}^n = \Omega^m{}_n$  expresses the reality of  $\Omega$ , whereas  $\Omega_m{}^n = -\Omega_n{}^m$  expresses its antisymmetry. For a symmetric matrix (such as  $I_4$  corresponding to  $\delta_m{}^n$ ) there is no need to distinguish between left and right indices, and so we write simply  $\delta_m^n$ .

Note also that  $\Omega_m{}^m = 0$ , which is frequently used in what follows.

### 2.3 LC supersymmetry transformations: superfield form

The component fields enter the superfield  $\phi$  in such a way that [17]

$$\begin{aligned}\phi| &= \frac{1}{\partial^+} A, & d_m \phi| &= \frac{i}{\partial^+} \chi_m, & d_{mn} \phi| &= -i\sqrt{2} C_{mn} \\ d_{mnk} \phi| &= -\sqrt{2} \varepsilon_{mnkl} \chi^l, & d_{mnkl} \phi| &= 2\varepsilon_{mnkl} \partial^+ \bar{A}.\end{aligned}\quad (2.19)$$

The superfield  $\phi$  is chiral,  $d^m \phi = 0$ , and satisfies the reality condition (“inside-out constraint”)

$$\bar{\phi} \equiv \phi^* = \frac{d_{[4]}}{2\partial^{+2}} \phi, \quad d_{[4]} \equiv d_1 d_2 d_3 d_4. \quad (2.20)$$

The same constraints must then be satisfied by the supersymmetry variation superfield  $\delta_\epsilon \phi$ . For the kinematical supersymmetry transformations, the answer is

$$\delta_{\epsilon_-} \phi = \sqrt{2}(\alpha^m q_m - \alpha_m q^m) \phi. \quad (2.21)$$

It is easy to verify that this reproduces (2.16) using that

$$\{q^m, q_n\} = Z\delta_n^m, \quad \{d^m, d_n\} = -Z\delta_n^m, \quad Z \equiv i\sqrt{2}\partial^+, \quad (2.22)$$

and that the  $\theta = 0$  projection (denoted by “|”) of  $q$ ’s is equal to the projection of  $d$ ’s. For the dynamical supersymmetry, we write

$$\delta_{\epsilon_+} \phi = (\eta^m Q_m - \eta_m Q^m) \phi, \quad (2.23)$$

where  $Q$ ’s must have a part linear in  $q$ ’s and a part cubic in  $q$ ’s. The superfield transformation should reproduce (2.17) upon projection. We already know the answer for the  $M$ -independent part of  $Q$ ’s [17]. For the  $M$ -dependent part, we choose the most general ansatz and then fix the coefficients accordingly. The final result of this analysis is

$$\begin{aligned}Q_m &= +i\frac{\partial}{\partial^+} q_m - \frac{M}{Z\partial^+} (Z\Omega_m{}^n q_n + \Omega_m{}^n q^k q_{nk} - \Omega_k{}^n q^k q_{nm}) \\ Q^m &= -i\frac{\partial}{\partial^+} q^m + \frac{M}{Z\partial^+} (Z\Omega_n{}^m q^n + \Omega_n{}^m q_k q^{nk} - \Omega_n{}^k q_k q^{nm}).\end{aligned}\quad (2.24)$$

where  $q_{nk} \equiv q_n q_k$ . Direct evaluation then yields

$$\{Q^m, Q_n\} = i\sqrt{2}\frac{1}{\partial^+} (\partial^2 - M^2) \delta_n^m, \quad (2.25)$$

which confirms the correctness of the result. The terms quartic in  $q$ ’s cancel thanks to the following identity

$$\begin{aligned}&\left\{ \delta_n^m (\Omega_k{}^l \Omega_t{}^s q_{ls} - q_{kt}) + 2\Omega_k{}^m (\Omega_t{}^s q_{sn} - \Omega_n{}^s q_{st}) \right. \\ &\quad \left. + 2\Omega_n{}^m \Omega_k{}^s q_{st} + 2\delta_t^m (\Omega_n{}^l \Omega_k{}^s q_{ls} - q_{nk}) \right\} q^{kt} = 0,\end{aligned}\quad (2.26)$$

which is the expanded version of the obvious identity

$$\delta_{[n}^m \Omega_k{}^l \Omega_t{}^s q_{ls]} q^{kt} = 0. \quad (2.27)$$

## 2.4 $SO(4) \times SO(4)$ $R$ -symmetry

When  $M = 0$ , the (anti)commutator of the dynamical and kinematical supersymmetry generators yields only the translation in the transverse direction. When  $M \neq 0$ , we also find terms quadratic in  $q$ 's, which can be organized in terms of the  $SO(8)$   $R$ -symmetry group generators (of the  $M = 0$  theory) [18]

$$\begin{aligned} T_n^m &= -\frac{1}{Z}(q^m q_n - \frac{1}{4}\delta_n^m q^k q_k), \quad T = -\frac{1}{4Z}(q^k q_k - q_k q^k) \\ T^{mn} &= -\frac{1}{Z}q^m q^n, \quad T_{mn} = -\frac{1}{Z}q_m q_n. \end{aligned} \quad (2.28)$$

Explicitly, we find that

$$\begin{aligned} \{q^m, Q_n\} &= -\sqrt{2}\delta_n^m \partial + i\sqrt{2}MS_n^m \\ \{q_m, Q_n\} &= i\sqrt{2}MS_{mn}, \quad \{q^m, Q^n\} = -i\sqrt{2}MS^{mn}, \end{aligned} \quad (2.29)$$

where we defined

$$\begin{aligned} S_n^m &= T_k^m \Omega_n^k + T_n^k \Omega_k^m - \delta_n^m T_l^k \Omega_k^l - T \Omega_n^m \\ S_{mn} &= \Omega_m^k T_{nk} - \Omega_n^k T_{mk}, \quad S^{mn} = \Omega_k^m T^{nk} - \Omega_k^n T^{mk}. \end{aligned} \quad (2.30)$$

We will now show that these generators generate the  $SO(4) \times SO(4)$  subgroup of  $SO(8)$ . We will see that  $S_m^n$  contains 8 independent hermitian generators of which two (the trace,  $iS_m^m$ , and the  $\Omega$ -trace,  $\Omega_m^n S_n^m$ ) commute with the other six. These six generators form one of  $SO(4)$ 's, whereas the two trace generators combine with 4 independent hermitian generators inside  $S_{mn}$  and  $S^{mn}$  to form another  $SO(4)$ . To see this, we first introduce parameters  $\omega^{mn}$ ,  $\omega_{mn} = (\omega^{mn})^*$  and  $\alpha_m^n = (\alpha_n^m)^*$  for the  $R$ -symmetry transformations, and define

$$\delta_\omega \phi = -\frac{1}{2}(\omega^{mn} S_{mn} - \omega_{mn} S^{mn})\phi, \quad \delta_\alpha \phi = \alpha_m^n S_n^m \phi. \quad (2.31)$$

More explicitly,

$$\begin{aligned} \delta_\omega \phi &= \frac{1}{Z}(\omega^{mn} \Omega_m^k q_{nk} - \omega_{mn} \Omega_k^m q^{nk})\phi \\ \delta_\alpha \phi &= \alpha_m^n \left[ \Omega_n^m \left( -1 + \frac{1}{Z}q^k q_k \right) + \delta_n^m \left( \frac{1}{Z}\Omega_k^l q^k q_l \right) - \frac{1}{Z}(\Omega_n^k q^m q_k + \Omega_k^m q^k q_n) \right] \phi. \end{aligned} \quad (2.32)$$

The six 't Hooft matrices,  $(\eta_a)_{mn}$  and  $(\tilde{\eta}_a)_{mn}$  (see Appendix A), form a basis in the space of antisymmetric 4 by 4 matrices, and so we write

$$\begin{aligned} \omega^{mn} &= \omega_1 \eta_1 + \omega_2 \eta_2 + \omega_3 \eta_3 + \tilde{\omega}_1 \tilde{\eta}_1 + \tilde{\omega}_2 \tilde{\eta}_2 + \tilde{\omega}_3 \tilde{\eta}_3 \\ \omega_{mn} &= \omega_1^* \eta_1 + \omega_2^* \eta_2 + \omega_3^* \eta_3 + \tilde{\omega}_1^* \tilde{\eta}_1 + \tilde{\omega}_2^* \tilde{\eta}_2 + \tilde{\omega}_3^* \tilde{\eta}_3. \end{aligned} \quad (2.33)$$

Conveniently enough, the products of 't Hooft matrices,  $(\eta_a \tilde{\eta}_b)_{mn}$ , together with the unit matrix  $I_4 = \delta_{mn}$  form a basis for all symmetric 4 by 4 matrices. This allows us to represent the hermitian matrix  $\alpha_m^n$  as

$$\begin{aligned} \alpha_m^n &= a_0 I_4 + a_{11} \eta_1 \tilde{\eta}_1 + a_{22} \eta_2 \tilde{\eta}_2 + a_{33} \eta_3 \tilde{\eta}_3 \\ &\quad + a_{12} \eta_1 \tilde{\eta}_2 + a_{13} \eta_1 \tilde{\eta}_3 + a_{21} \eta_2 \tilde{\eta}_1 + a_{23} \eta_2 \tilde{\eta}_3 + a_{31} \eta_3 \tilde{\eta}_1 + a_{32} \eta_3 \tilde{\eta}_2 \\ &\quad + i(a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3 + \tilde{a}_1 \tilde{\eta}_1 + \tilde{a}_2 \tilde{\eta}_2 + \tilde{a}_3 \tilde{\eta}_3), \end{aligned} \quad (2.34)$$



where in the first, second and third lines we have, respectively, the diagonal, symmetric off-diagonal and antisymmetric matrices. All 16 coefficients  $a$ 's are real, whereas the 6 coefficients  $\omega$ 's are complex. However, many of them drop out from  $\delta_\omega\phi$  and  $\delta_\alpha\phi$  in (2.32) for symmetry reasons. With  $\Omega_m^n = (\eta_3)_{mn}$ , we find that only

$$\omega_1, \quad \omega_2; \quad a_0, \quad a_3, \quad \tilde{a}_1, \quad \tilde{a}_2, \quad \tilde{a}_3, \quad a_{31}, \quad a_{32}, \quad a_{33} \quad (2.35)$$

contribute. This constitutes 12 real parameters, which matches the dimension of  $SO(4) \times SO(4)$ . More explicitly, substituting (2.33) and (2.34) into (2.32), we find that

$$\begin{aligned} \delta_\omega\phi &= \left( -\omega_1 \overline{W}_2 - \omega_1^* W_2 + \omega_2 \overline{W}_1 + \omega_2^* W_1 \right) \phi \\ \delta_\alpha\phi &= 2 \left[ a_0 V + a_{3a} B_a - i(a_3 U + \tilde{a}_a A_a) \right] \phi, \end{aligned} \quad (2.36)$$

where  $a = 1, 2, 3$  and we defined

$$\begin{aligned} W_a &= (\eta_a)_{mn} \frac{1}{Z} q^{mn}, \quad \overline{W}_a = (\eta_a)_{mn} \frac{1}{Z} q_{mn} \\ U &= -2 + \frac{1}{Z} q^k q_k, \quad V = \Omega_m^n \frac{1}{Z} q^m q_n \\ A_a &= (\eta_3 \tilde{\eta}_a)_{mn} \frac{1}{Z} q^m q_n, \quad B_a = (\tilde{\eta}_a)_{mn} \frac{1}{Z} q^m q_n. \end{aligned} \quad (2.37)$$

The hermiticity (or complex conjugation [17]) properties of these generators are

$$U^* = U, \quad V^* = -V, \quad W_a^* = -\overline{W}_a, \quad A_a^* = A_a, \quad B_a^* = -B_a. \quad (2.38)$$

Reorganizing them into the following four triplets of *hermitian* generators

$$\begin{aligned} X_1 &= -\frac{1}{8}(W_2 - \overline{W}_2) + \frac{i}{8}(W_1 + \overline{W}_1), & Y_1 &= \frac{1}{8}(W_2 - \overline{W}_2) + \frac{i}{8}(W_1 + \overline{W}_1) \\ X_2 &= +\frac{1}{8}(W_1 - \overline{W}_1) + \frac{i}{8}(W_2 + \overline{W}_2), & Y_2 &= \frac{1}{8}(W_1 - \overline{W}_1) - \frac{i}{8}(W_2 + \overline{W}_2) \\ X_3 &= \frac{1}{4}(U - iV), & Y_3 &= \frac{1}{4}(U + iV) \\ R_a &= -\frac{1}{2}(A_a + iB_a), & L_a &= \frac{1}{2}(A_a - iB_a), \end{aligned} \quad (2.39)$$

we find that

$$[X_a, X_b] = i\varepsilon_{abc} X_c, \quad (2.40)$$

and similarly for  $Y_a$ ,  $R_a$  and  $L_a$ , whereas all other commutators vanish. This proves that the  $R$ -symmetry group of the mass-deformed theory is  $SU(2) \times SU(2) \times SU(2) \times SU(2)$ , which is the same as  $SO(4) \times SO(4)$ .

On another hand, it is also instructive to see how the  $R$ -symmetry transformations act on the scalars. Projecting (2.32) to find the corresponding variations of  $A$  and  $C_{mn}$ , and using (2.19) together with (2.12), we find after a little algebra that

$$\begin{aligned} \delta_R X^3 &= 4(-a_3 X^4 - b_2 X^5 + b_1 X^6), & \delta_R X^7 &= 4(-\tilde{a}_1 X^8 - \tilde{a}_2 X^9 - \tilde{a}_3 X^{10}) \\ \delta_R X^4 &= 4(a_3 X^3 - c_2 X^5 + c_1 X^6), & \delta_R X^8 &= 4(\tilde{a}_1 X^7 + a_{33} X^9 - a_{32} X^{10}) \\ \delta_R X^5 &= 4(b_2 X^3 + c_2 X^4 + a_0 X^6), & \delta_R X^9 &= 4(\tilde{a}_2 X^7 - a_{33} X^8 + a_{31} X^{10}) \\ \delta_R X^6 &= 4(-b_1 X^3 - c_1 X^4 - a_0 X^5), & \delta_R X^{10} &= 4(\tilde{a}_3 X^7 + a_{32} X^8 - a_{31} X^9), \end{aligned} \quad (2.41)$$

where  $\delta_R = \delta_\omega + \delta_\alpha$ , and we defined  $\omega_{1,2} = b_{1,2} + ic_{1,2}$ . This clearly shows the  $SO(4) \times SO(4)$  structure of the surviving  $R$ -symmetry transformations.

## 2.5 Hamiltonian as a quadratic form

The on-shell Lagrangian, obtained by substituting (2.9) into (2.1), can be easily transformed into the superfield form,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}X^I(\square - M^2)X^I - \frac{i}{2}\bar{\Psi}_+\Gamma_- \frac{1}{2\partial^+}(\square - M^2)\Psi_+ \\ &= \bar{A}(\square - M^2)A + \frac{1}{4}C_{mn}(\square - M^2)C^{mn} + \frac{i}{\sqrt{2}}\chi_m \frac{1}{\partial^+}(\square - M^2)\chi^m \\ &= -\frac{1}{8}\int d^4\theta d^4\theta \left\{ \bar{\phi} \frac{1}{\partial^{+2}}(\square - M^2)\phi \right\},\end{aligned}\tag{2.42}$$

where we used that

$$\int d^4\theta d^4\bar{\theta}(\dots) = d^{[4]}d_{[4]}(\dots)_| = \frac{1}{4!}\varepsilon_{ijkl}d^{ijkl}\frac{1}{4!}\varepsilon^{mnpq}d_{mnpq}(\dots)_|.\tag{2.43}$$

The LC Hamiltonian (defined with respect to the LC “time”  $x^+$ ) is then <sup>4</sup>

$$\mathcal{H} \equiv \frac{\delta\mathcal{L}}{\delta(\partial^-\phi)}\partial^-\phi - \mathcal{L} = \frac{1}{8}\int d^4\theta d^4\theta \left\{ \bar{\phi} \frac{1}{\partial^{+2}}(\partial^2 - M^2)\phi \right\},\tag{2.44}$$

where we used that  $\square = \partial^2 - 2\partial^+\partial^-$ . We claim that this can be rewritten as a quadratic form in the dynamical supersymmetry transformations,

$$\mathcal{H} = \frac{i}{16\sqrt{2}}\int d^4\theta d^4\theta \left\{ (Q_m\bar{\phi}) \frac{1}{\partial^+}(Q^m\phi) \right\}.\tag{2.45}$$

The proof is the same as in [15], because, thanks to the following identity

$$2Z\Omega_m{}^n q_n + \Omega_m{}^n [q^k, q_{nk}] + \Omega_k{}^n [q_{nm}, q^k] = 0,\tag{2.46}$$

we can simply integrate by parts with  $Q_m$ . Using (2.20), we then find that

$$\mathcal{H} = -\frac{i}{32\sqrt{2}}\int d^4\theta d^4\bar{\theta} \left( \bar{\phi} \frac{1}{\partial^+} \{Q_m, Q^m\} \phi \right),\tag{2.47}$$

after which we use (2.25) to confirm the quadratic form property of the Hamiltonian.

## 2.6 Dynamical Lorentz boost

When  $M = 0$ , conformal invariance allows the dynamical boost generator,  $\mathcal{J}^-$ , to be calculated by commuting the Hamiltonian shift generator,  $\mathcal{P}^-$ , with the kinematical generator of special conformal transformations,  $K^+$  [18]. Once  $M \neq 0$ , the conformal invariance is broken and  $\mathcal{J}^-$

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<sup>4</sup>  $\mathcal{H}$  should really be called the Hamiltonian density. The Hamiltonian is  $H = \int dx^- dx^2 \mathcal{H}$ . We will work with  $\mathcal{H}$  while freely dropping total  $\partial^+ = -\partial_-$  and  $\partial = \partial_2$  derivatives, which is justified once we go back to  $H$ .

has to be derived from scratch. To do so, we start from the covariant form of the Lorentz transformations (2.3). Defining  $\delta_{\mathcal{J}-}$  as the part of  $\delta_\lambda$  multiplied by  $\lambda_{2-}$ , we find that

$$\begin{aligned}\delta_{\mathcal{J}-} X^I &= (x\partial^- - x^-\partial^+) X^I \\ \delta_{\mathcal{J}-} \Psi_+ &= (x\partial^- - x^-\partial^+) \Psi_+ + \frac{1}{2\partial^+} (\partial + M\Gamma_2\Gamma_{3456}) \Psi_+ ,\end{aligned}\tag{2.48}$$

where as usual we set  $x^+ = 0$  and substituted (2.9) for  $\Psi_-$ . Reducing this to the variation of the superfield component fields, we obtain

$$\begin{aligned}\delta_{\mathcal{J}-} A &= (x\partial^- - x^-\partial) A \\ \delta_{\mathcal{J}-} \chi_m &= \left(x\partial^- - x^-\partial + \frac{\partial}{2\partial^+}\right) \chi_m + \frac{iM}{2\partial^+} \Omega_m{}^n \chi_n \\ \delta_{\mathcal{J}-} C_{mn} &= (x\partial^- - x^-\partial) C_{mn} .\end{aligned}\tag{2.49}$$

The superfield expression for the  $M$ -independent part of  $\delta_{\mathcal{J}-}\phi$  follows easily by comparison with [18]. To find the corresponding expression for the  $O(M)$  part, we observe that, with the  $R$ -symmetry generators  $U$  and  $V$  defined in (2.37), we have

$$\begin{aligned}iU(A, \chi_m, C_{mn}, \chi^m, \bar{A}) &= (-2iA, -i\chi_m, 0, i\chi^m, 2i\bar{A}) \\ V(A, \chi_m, C_{mn}, \chi^m, \bar{A}) &= (0, \Omega_m{}^n \chi_n, \Omega_m{}^k C_{kn} - \Omega_n{}^k C_{km}, \Omega_n{}^m \chi^n, 0) ,\end{aligned}\tag{2.50}$$

and therefore

$$iUV(A, \chi_m, C_{mn}, \chi^m, \bar{A}) = (0, -i\Omega_m{}^n \chi_n, 0, -i\Omega_n{}^m \chi^n, 0) .\tag{2.51}$$

It then follows that

$$\delta_{\mathcal{J}-}\phi = \left(x\partial^- - x^-\partial + \left(\frac{1}{2}\mathcal{N} - 1\right)\frac{\partial}{\partial^+}\right)\phi - \frac{iM}{2\partial^+} UV\phi ,\tag{2.52}$$

where  $\mathcal{N} = \theta^m \partial_m + \theta_m \partial^m = 4 + \frac{1}{Z}(q^m d_m + q_m d^m)$  [18] and

$$\partial^- = \frac{1}{2\partial^+} (\partial^2 - M^2) .\tag{2.53}$$

We thus have found the mass deformation of all the (non-conformal) dynamical generators of the (abelian) BLG theory. The kinematical (non-conformal) generators receive no  $M$ -dependent modifications, and are the same as in [18]. The conformal symmetries (dilatations, special conformal and superconformal) are, obviously, broken by the mass deformation. In the next section, we will discuss the non-abelian generalization of these results.

### 3 The full mass-deformed BLG theory

In this section, we extend the preceding results to the case when  $f^{bcd}_a$  is non-zero. The mass deformation of the supersymmetry transformations has no terms with  $M$  multiplying  $f^{bcd}_a$ , and so (2.23) generalizes trivially. The  $R$ -symmetry generators are all kinematical, and receive no modifications from the  $f$ 's. The difficult part is to verify that the full LC superspace Hamiltonian can still be written as a quadratic form in the dynamical supersymmetry transformations. We will show that this is, indeed, the case.

#### 3.1 Supersymmetry transformations

The complete supersymmetry transformations of the mass-deformed BLG theory are [4, 5, 22]

$$\begin{aligned}\delta_\epsilon X_a^I &= i\bar{\epsilon}\Gamma^I\Psi_a \\ \delta_\epsilon\Psi_a &= D_\mu X_a^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X_b^IX_c^JX_d^Kf^{bcd}_a\Gamma^{IJK}\epsilon - M\Gamma_{3456}\Gamma^IX_a^I\epsilon \\ \delta_\epsilon\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma^IX_c^I\Psi_af^{cdb}_a, \end{aligned}\tag{3.1}$$

where  $f^{bcd}_a$  are totally antisymmetric in the upper indices, and satisfy the Fundamental Identity [23]

$$f^{[efg}{}_df^{c]db}_a = 0. \tag{3.2}$$

The commutator of these transformations closes into the translation, gauge transformation and an  $M$ -dependent  $R$ -symmetry transformation, plus terms proportional to the equations of motion. To derive the LC superspace transformation laws, we need to go through the following steps [17]

- fix the LC gauge  $\tilde{A}_-{}^b{}_a = 0$ ;
- solve equations of motion for dependent field components ( $\tilde{A}_+{}^b{}_a$ ,  $\tilde{A}_2{}^b{}_a$  and  $\Psi_{a-}$ );
- modify supersymmetry transformations by adding compensating gauge transformations required to stay in the gauge;
- find the modified supersymmetry transformations of  $A_a$ ,  $C_{mna}$  and  $\chi_{ma}$ ;
- guess and confirm the corresponding superfield transformation law.

As the equation of motion for  $\tilde{A}_\mu{}^b{}_a$  is not affected by the mass-deformation, the expressions for  $\tilde{A}_+{}^b{}_a$  and  $\tilde{A}_2{}^b{}_a$  remain the same as in [17]. As there are no  $M$ -dependent corrections to  $\delta_\epsilon\tilde{A}_\mu{}^b{}_a$ , the parameters of the compensating gauge transformations  $\tilde{\Lambda}^b{}_a$  are also the same as in [17]. The equation of motion for  $\Psi_a$  is

$$\Gamma^\mu D_\mu\Psi_a + \frac{1}{2}\Gamma^{IJ}\Psi_bX_c^IX_d^Jf^{bcd}_a + M\Gamma_{3456}\Psi_a = 0, \tag{3.3}$$

and so the expression for  $\Psi_{a-}$  is modified. However, as the  $M$ -dependent term comes without  $f^{bcd}_a$ , its effect on the supersymmetry transformation of  $\phi_a$  is exactly the same as in the  $f^{bcd}_a = 0$

case considered in Section 2.<sup>5</sup> Therefore, combining (2.23) with the results of [17], we conclude that the full LC superspace transformation laws in the mass-deformed BLG theory are

$$\begin{aligned}\delta_{\epsilon_-}\phi_a &= \sqrt{2}(\alpha^m q_m - \alpha_m q^m)\phi_a \\ \delta_{\epsilon_+}\phi_a &= (\beta^m Q_m - \beta_m Q^m)\phi_a + i\beta_m W_a^m + \frac{d^{[4]}}{2\partial^+} (i\beta^m W_{ma}) ,\end{aligned}\quad (3.4)$$

where  $Q$ 's are given in (2.24), and

$$\begin{aligned}W_a^m &= -\frac{i}{3\sqrt{2}}\varepsilon^{mnkl}\frac{1}{\partial^+}\left(\partial^+\phi_b\cdot\frac{1}{\partial^+}(\partial^+\phi_c\cdot q_{nkl}\phi_d + 3\partial^+q_n\phi_c\cdot q_{kl}\phi_d)\right)f^{bcd}_a \\ W_{ma} &= -\frac{i}{3\sqrt{2}}\varepsilon_{mnkl}\frac{1}{\partial^+}\left(\partial^+\bar{\phi}_b\cdot\frac{1}{\partial^+}(\partial^+\bar{\phi}_c\cdot q^{nkl}\bar{\phi}_d + 3\partial^+q^n\bar{\phi}_c\cdot q^{kl}\bar{\phi}_d)\right)f^{bcd}_a .\end{aligned}\quad (3.5)$$

Note that we chose the form of  $W$ 's that involves  $q$ 's instead of  $d$ 's (see (7.43) in [17]). As we will see, with this choice the proof of the quadratic form property of the LC superspace Hamiltonian simplifies tremendously (cf. [18]).

### 3.2 The Lagrangian and the LC Hamiltonian

In order to write down the Lagrangian invariant under the supersymmetry transformations (3.1), we need a metric  $h_{ab}$  for raising and lowering the gauge indices. Requiring that the resulting  $f^{abcd} \equiv f^{abc}{}_e h^{ed}$  is totally antisymmetric, the Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{BLG} + \mathcal{L}_M \\ \mathcal{L}_{BLG} &= -\frac{1}{2}(D_\mu X_a^I)(D^\mu X_a^I) + \frac{i}{2}\bar{\Psi}_a\Gamma^\mu D_\mu\Psi_a + \frac{i}{4}\bar{\Psi}_b\Gamma_{IJ}X_c^IX_d^J\Psi_af^{bcda} \\ &\quad + \frac{1}{2}\varepsilon^{\mu\nu\lambda}A_{\mu ab}(\partial_\nu\tilde{A}_\lambda^{ab} + \frac{2}{3}\tilde{A}_\nu{}^a{}_c\tilde{A}_\lambda^{cb}) - \frac{1}{12}f^{abcd}f^{efg}{}_dX_a^IX_b^JX_c^KX_e^IX_f^JX_g^K \\ \mathcal{L}_M &= -\frac{M^2}{2}X_a^IX_a^I + \frac{i}{2}M\bar{\Psi}_a\Gamma_{3456}\Psi_a - 4Mf^{abcd}(X_a^3X_b^4X_c^5X_d^6 + X_a^7X_b^8X_c^9X_d^{10}) ,\end{aligned}\quad (3.6)$$

where  $D_\mu\Psi_a = \partial_\mu\Psi_a - \tilde{A}_\mu{}^b{}_a\Psi_b$  with  $\tilde{A}_\mu{}^b{}_a = f^{cdb}{}_aA_{\mu cd}$ . In the LC gauge, with all the dependent fields substituted into the Lagrangian (as in [19] but in the conventions of [17]), the  $\Psi$ -independent part of the Lagrangian reduces to  $\mathcal{L}_X = -X_a^I\partial^+\partial^-X_a^I - \mathcal{H}_X$  where

$$\begin{aligned}\mathcal{H}_X &= -\frac{1}{2}X_a^I(\partial^2 - M^2)X_a^I - f^{abcd}(X_a^I\partial X_b^I)\frac{1}{\partial^+}(X_c^J\partial^+X_d^J) \\ &\quad + \frac{1}{2}f^{abcd}f^{ab'c'd'}(X_b^IX_{b'}^I)\cdot\frac{1}{\partial^+}(X_c^J\partial^+X_d^J)\cdot\frac{1}{\partial^+}(X_{c'}^KX_{d'}^K) \\ &\quad + \frac{1}{12}f^{abcd}f^{ab'c'd'}(X_b^IX_{b'}^I)(X_c^JX_{c'}^J)(X_d^JX_{d'}^J) \\ &\quad + 4Mf^{abcd}(X_a^3X_b^4X_c^5X_d^6 + X_a^7X_b^8X_c^9X_d^{10}) .\end{aligned}\quad (3.7)$$

The first three lines are  $SO(8)$  invariant and can be rewritten in terms of  $A$ 's and  $C$ 's using

$$X^IX^I = A\bar{A}' + \bar{A}A' + \frac{1}{2}C_{mn}C'^{mn} .\quad (3.8)$$

The  $O(M)$  part of the LC Hamiltonian breaks  $SO(8)$  down to  $SO(4) \times SO(4)$ . Its form in terms of  $A$ 's and  $C$ 's is given in equation (C.20) of Appendix C.

<sup>5</sup> Similar analysis shows that our result (2.52) captures all the  $M$ -dependence of the dynamical Lorentz boost,  $\delta_{\mathcal{J}^-}\phi_a$ . Note that  $\partial^-\phi_a$  there receives an additional  $O(M)$  correction, which can be deduced from the full Hamiltonian.

### 3.3 Hamiltonian as a quadratic form

According to (3.4), we have  $\delta_{\epsilon_+}\phi_a = \delta_{\bar{\beta}\mathcal{Q}}\phi_a + \delta_{\beta\bar{\mathcal{Q}}}\phi_a$  where

$$\begin{aligned}\delta_{\bar{\beta}\mathcal{Q}}\phi_a &= \beta_m(-Q^m\phi_a + iW_a^m), & \delta_{\beta\bar{\mathcal{Q}}}\phi_a &= \frac{d^{[4]}}{2\partial^+}(\delta_{\bar{\beta}\mathcal{Q}}\phi_a)^* \\ (\delta_{\bar{\beta}\mathcal{Q}}\phi_a)^* &= \beta^m(Q_m\bar{\phi}_a + iW_{ma}).\end{aligned}\tag{3.9}$$

We therefore expect that the quadratic form property of the Hamiltonian with  $f^{abcd} = 0$ , see (2.45), generalizes to

$$\mathcal{H} = \frac{i}{16\sqrt{2}} \int d^4\theta d^4\bar{\theta} (Q_m\bar{\phi}_a + iW_{ma}) \frac{1}{\partial^+} (Q^m\phi_a - iW_a^m). \tag{3.10}$$

Expanding in powers of  $f$ 's, we have  $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)}$ , where

$$\begin{aligned}\mathcal{H}^{(0)} &= \frac{i}{16\sqrt{2}} \int d^4\theta d^4\bar{\theta} (Q_m\bar{\phi}_a) \frac{1}{\partial^+} (Q^m\phi_a) \\ \mathcal{H}^{(1)} &= -\frac{1}{16\sqrt{2}} \int d^4\theta d^4\bar{\theta} (Q^m\phi_a) \frac{1}{\partial^+} W_{ma} + c.c. \\ \mathcal{H}^{(2)} &= \frac{i}{16\sqrt{2}} \int d^4\theta d^4\bar{\theta} W_{ma} \frac{1}{\partial^+} W_a^m.\end{aligned}\tag{3.11}$$

We have already verified that  $\mathcal{H}^{(0)}$  reproduces the corresponding part in (3.7). In Appendices C and D, we show that the same is true for the  $O(f)$  and  $O(f^2)$  parts. Kinematical supersymmetry then guarantees that the  $\Psi$ -dependent part of  $\mathcal{H}$  is reproduced correctly as well. The quadratic form property of the LC superspace Hamiltonian is therefore rigorously established.

## 4 Summary and discussion

In this paper, we have analyzed the mass-deformed BLG theory [4, 5] from the LC superspace point of view [12]. We found that the mass deformation is treated as an interaction in the sense that the (surviving) kinematical symmetry generators [18], including the kinematical supersymmetry generators  $q$ 's, are not modified by it.<sup>6</sup> The (surviving) dynamical symmetry generators, the dynamical supersymmetries  $\mathcal{Q}$ 's, the Hamiltonian shift  $\mathcal{P}^-$ , and the dynamical Lorentz boost  $\mathcal{J}^-$ , all receive  $M$ -dependent corrections.

The modification of the  $\mathcal{Q}$ 's is the simplest, but non-trivial: it is linear in  $M$ , independent of  $f^{bcd}_a$ , proportional to  $\Omega_m{}^n$  and cubic in the  $q$ 's. The matrix  $\Omega$  carries  $SU(4)$  indices and specifies a quaternionic direction in the algebra of  $SO(4) \subset SU(4)$ . It plays the key role in reducing the  $SO(8)$   $R$ -symmetry of the BLG theory down to  $SO(4) \times SO(4)$ .

The  $\mathcal{P}^-$  can be determined either from the commutator of two  $\mathcal{Q}$ 's or as a functional derivative of the LC superspace Hamiltonian  $H$  [18]. The resulting expression is complicated, and has both  $O(M)$  and  $O(M^2)$  parts. However,  $H$  itself is extremely simple, being given by the quadratic form (3.10).

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<sup>6</sup> Other kinematical generators, including the conformal ones and part of the  $SO(8)$   $R$ -symmetry, cease to represent symmetries when the mass  $M$  is introduced.

The  $M$ -independent part of  $\mathcal{J}^-$  can be derived by commuting  $\mathcal{P}^-$  with the kinematical special conformal generator  $K^+$  [18]. Most of its  $M$ -dependence then comes from adjusting the value of  $\partial^-$ , encoded in the Hamiltonian. Its remaining  $O(M)$  part, see (2.52), turns out to be given by a single term quadratic in the surviving  $R$ -symmetry generators.

The quadratic form property of the LC superspace Hamiltonian  $H$  has been established via an explicit calculation. We extended the proof of [18] to the quadratic order in  $f^{abcd}$ , as well as proved that the same quadratic form correctly describes the  $M$ -dependent parts of  $H$ . Still lacking, however, is the fundamental understanding of this property (first observed in [15]). It appears to be rooted into maximal supersymmetry, which in turn imposes the reality (“inside-out”) constraint (2.20) on the superfield  $\phi$  [12] and leads to the quadratic form property at the  $O(f^0)$  level. Presumably, the preservation of this property while turning on the structure constants  $f$ ’s can be attributed to analyticity of extended supersymmetry [24].

The mass deformation of the BLG theory serves as a supersymmetry preserving IR regulator, and therefore we expect that the analysis performed in this paper should be useful for studying quantum properties of the BLG theory in LC superspace.

It would also be very interesting to understand the LC superspace formulation of theories with less-than-maximal supersymmetry (such as [25, 26]). In particular, to study deviations from the quadratic form property of the LC superspace Hamiltonians in these cases.

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## A Gamma matrices and ’t Hooft symbols

The representation of the  $d = 11$  gamma matrices that we use is as follows [17]

$$\begin{aligned}
\Gamma_0 &= -i(I_2 \otimes \sigma_1) \otimes (I_4 \otimes I_2), & \Gamma_5 &= -i(I_2 \otimes \sigma_3) \otimes (\eta_1 \otimes \sigma_1) \\
\Gamma_1 &= i(\sigma_3 \otimes i\sigma_2) \otimes (I_4 \otimes I_2), & \Gamma_6 &= -i(I_2 \otimes \sigma_3) \otimes (\eta_2 \otimes \sigma_1) \\
\Gamma_2 &= (I_2 \otimes \sigma_3) \otimes (I_4 \otimes \sigma_3), & \Gamma_7 &= -i(I_2 \otimes \sigma_3) \otimes (\eta_3 \otimes \sigma_1) \\
\Gamma_3 &= i(\sigma_1 \otimes i\sigma_2) \otimes (I_4 \otimes I_2), & \Gamma_8 &= -(I_2 \otimes \sigma_3) \otimes (\tilde{\eta}_1 \otimes i\sigma_2) \\
\Gamma_4 &= i(\sigma_2 \otimes i\sigma_2) \otimes (I_4 \otimes I_2), & \Gamma_9 &= -(I_2 \otimes \sigma_3) \otimes (\tilde{\eta}_2 \otimes i\sigma_2) \\
& & \Gamma_{10} &= -(I_2 \otimes \sigma_3) \otimes (\tilde{\eta}_3 \otimes i\sigma_2),
\end{aligned}$$

where the Pauli matrices  $\sigma_a$ ,  $a = 1, 2, 3$ , are standard

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.1})$$

and the 't Hooft symbols  $\eta_{amn}$  and  $\tilde{\eta}_{amn}$  are given by

$$\begin{aligned}
\eta_1 = +\sigma_1 \otimes i\sigma_2 &= \begin{pmatrix} & & + \\ & + & \\ - & - & \end{pmatrix} & \tilde{\eta}_1 = -i\sigma_2 \otimes \sigma_1 &= \begin{pmatrix} & & - \\ & + & \\ + & - & \end{pmatrix} \\
\eta_2 = -\sigma_3 \otimes i\sigma_2 &= \begin{pmatrix} & - & \\ + & & + \\ & - & \end{pmatrix} & \tilde{\eta}_2 = -I_2 \otimes i\sigma_2 &= \begin{pmatrix} & - & \\ + & & - \\ & + & \end{pmatrix} \\
\eta_3 = +i\sigma_2 \otimes I_2 &= \begin{pmatrix} + & & \\ - & & \\ & & + \\ & - & \end{pmatrix} & \tilde{\eta}_3 = +i\sigma_2 \otimes \sigma_3 &= \begin{pmatrix} + & & \\ - & & \\ & & - \\ & + & \end{pmatrix}. \quad (\text{A.2})
\end{aligned}$$

They satisfy  $\sigma_a \sigma_b = I_2 \delta_{ab} + i\varepsilon_{abc} \sigma_c$  and <sup>7</sup>

$$\eta_a \eta_b = -I_4 \delta_{ab} - \varepsilon_{abc} \eta_c, \quad \eta_a \tilde{\eta}_b = \tilde{\eta}_b \eta_a, \quad \tilde{\eta}_a \tilde{\eta}_b = -I_4 \delta_{ab} - \varepsilon_{abc} \tilde{\eta}_c. \quad (\text{A.3})$$

Note that  $\eta_1 \eta_2 \eta_3 = \tilde{\eta}_1 \tilde{\eta}_2 \tilde{\eta}_3 = I_4$ . The only independent products, therefore, are

$$\begin{aligned}
\eta_1 \tilde{\eta}_1 &= \begin{pmatrix} + & & \\ & - & \\ & & - \\ & & + \end{pmatrix}, & \eta_1 \tilde{\eta}_2 &= \begin{pmatrix} & + & \\ + & & \\ & & + \end{pmatrix}, & \eta_1 \tilde{\eta}_3 &= \begin{pmatrix} & & + \\ & - & \\ + & & - \end{pmatrix} \\
\eta_2 \tilde{\eta}_1 &= \begin{pmatrix} & + & \\ + & & \\ & - & \\ & & - \end{pmatrix}, & \eta_2 \tilde{\eta}_2 &= \begin{pmatrix} - & & \\ & + & \\ & & - \\ & & + \end{pmatrix}, & \eta_2 \tilde{\eta}_3 &= \begin{pmatrix} & & + \\ & + & \\ + & & + \end{pmatrix} \\
\eta_3 \tilde{\eta}_1 &= \begin{pmatrix} & & + \\ + & & \\ & + & \\ & + & \end{pmatrix}, & \eta_3 \tilde{\eta}_2 &= \begin{pmatrix} & & - \\ & + & \\ + & & \\ - & & \end{pmatrix}, & \eta_3 \tilde{\eta}_3 &= \begin{pmatrix} - & & \\ & - & \\ & & + \\ & & + \end{pmatrix}. \quad (\text{A.4})
\end{aligned}$$

The  $d = 11$  charge conjugation matrix is

$$C = i(\sigma_2 \otimes \sigma_3) \otimes (I_4 \otimes \sigma_1), \quad (\text{A.5})$$

and it is used to define the conjugated spinors  $\bar{\epsilon} = \epsilon^T C$  and  $\bar{\Psi} = \Psi^T C$ . We also observe that

$$\begin{aligned}
\Gamma_{012} &= -(\sigma_3 \otimes I_2) \otimes (I_4 \otimes \sigma_3) \\
\Gamma_{3456} &= i(\sigma_3 \otimes I_2) \otimes (\eta_3 \otimes I_2) \\
\Gamma_{789(10)} &= i(I_2 \otimes I_2) \otimes (\eta_3 \otimes \sigma_3), \quad (\text{A.6})
\end{aligned}$$

---

<sup>7</sup> The  $(I_4, \eta_a)$  and  $(I_4, \tilde{\eta}_a)$  are two representations of quaternions as  $SO(4)$  rotation matrices corresponding to left- and right-multiplication of quaternions, respectively (see e.g. [27]).



from which  $\Gamma_{789(10)} = -\Gamma_{3456}\Gamma_{012}$  follows. Finally, our conventions [17] are such that

$$\left\{ \mathcal{M}_4 \otimes \begin{pmatrix} a^m{}_n & b^{mn} \\ c_{mn} & d_m{}^n \end{pmatrix} \right\} \begin{pmatrix} \psi^n \\ \psi_n \end{pmatrix} = \begin{pmatrix} \mathcal{M}_4(a^m{}_n \psi^n + b^{mn} \psi_n) \\ \mathcal{M}_4(c_{mn} \psi^n + d_m{}^n \psi_n) \end{pmatrix}, \quad (\text{A.7})$$

where the 4 by 4 matrix  $\mathcal{M}_4$  acts on the (implicit) spinor indices of  $\psi$ 's.

## B Useful identities

The self-dual tensor  $C_{mn}$ , satisfying

$$(C_{mn})^* = C^{mn} = \frac{1}{2} \varepsilon^{mnkl} C_{kl}, \quad (\text{B.1})$$

enjoys many interesting identities. The basic identity that we need is

$$(C_{ik}, C^{jk}) + (C^{jk}, C_{ik}) = \frac{1}{2} (C_1, C_1) \delta_i^j, \quad (\text{B.2})$$

where  $(C_1, C_1) \equiv (C_{mn}, C^{mn}) = (C^{mn}, C_{mn})$ . Using the shorthand notation  $C_{12} \equiv C_{i_1 i_2}$ , etc., we find that

$$(\underline{C_{12}}, \underline{C^{23}}, C_{34}, C^{41}) + (C^{23}, C_{12}, C_{34}, C^{41}) = \frac{1}{2} (C_1, C_1, C_2, C_2), \quad (\text{B.3})$$

where we underlined the two  $C$ 's to which the identity (B.2) is applied. Noting that

$$\begin{aligned} & \left[ (\underline{C_{12}}, \underline{C^{23}}, C_{34}, C^{41}) + (C^{23}, C_{12}, C_{34}, C^{41}) \right] \\ & - \left[ (\underline{C^{23}}, C_{12}, \underline{C_{34}}, C^{41}) + (C_{34}, C_{12}, C^{23}, C^{41}) \right] \\ & + \left[ (C_{34}, \underline{C_{12}}, \underline{C^{23}}, C^{41}) + (C_{34}, C^{23}, C_{12}, C^{41}) \right] = 2(C_{12}, C^{23}, C_{34}, C^{41}), \end{aligned} \quad (\text{B.4})$$

we deduce the following identity

$$(C_{12}, C^{23}, C_{34}, C^{41}) = \frac{1}{4} \left\{ (C_1, C_1, C_2, C_2) - (C_1, C_2, C_1, C_2) + (C_1, C_2, C_2, C_1) \right\}. \quad (\text{B.5})$$

Denoting the LHS of (B.4) as “(12/23) – (23/34) + (12/23),” we find that the corresponding sequence needed to similarly reduce  $(C_{12}, C^{23}, C_{34}, C^{45}, C_{56}, C^{61})$  is

$$\begin{aligned} & (12/23) - (23/34) + (34/45) - (12/23) + (23/34) - (12/23) + (56/61) \\ & = (C_{12}, C^{23}, C_{34}, C^{45}, C_{56}, C^{61}) + (C^{45}, C_{34}, C^{23}, C_{12}, C^{61}, C_{56}), \end{aligned} \quad (\text{B.6})$$

which yields

$$\begin{aligned} (C_{12}, C^{23}, C_{34}, C^{45}, C_{56}, C^{61}) + c.c. &= \frac{1}{2} \left\{ (C_5, C_5, C_{12}, C^{23}, C_{34}, C^{41}) \right. \\ & - (C_5, C_{12}, C_5, C^{23}, C_{34}, C^{41}) + (C_5, C_{12}, C^{23}, C_5, C_{34}, C^{41}) \\ & - (C_{12}, C_5, C_5, C^{23}, C^{41}, C_{34}) + (C_{12}, C_5, C^{23}, C_5, C^{41}, C_{34}) \\ & \left. - (C_{12}, C^{23}, C_5, C_5, C^{41}, C_{34}) + (C_{12}, C^{23}, C_{34}, C^{41}, C_5, C_5) \right\}. \end{aligned} \quad (\text{B.7})$$

Now the identity (B.5) can be applied and we find 21 terms on the RHS, of which 6 terms cancel upon relabelling of indices. The remaining 15 terms combine in a particularly nice way if we order the  $C$ 's as follows

$$(C_{12}, C^{61}, C_{56}, C^{23}, C_{34}, C^{45}) + c.c. = -\frac{1}{8} \left\{ 522(511 + 115 - 151) \right. \\ \left. + (512 - 521)(512 + 125 - 152) + (112 - 121)(255 + 552 - 525) \right\}, \quad (\text{B.8})$$

where  $522511 \equiv (C_5, C_2, C_2, C_5, C_1, C_1)$ , etc.

## C The linear in $f^{abcd}$ part of the Hamiltonian

The  $O(f)$  part of the quadratic form Hamiltonian (3.10) is given by

$$\mathcal{H}^{(1)} = -\frac{1}{16\sqrt{2}} d^{[4]} \left\{ Q^m d_{[4]} \phi_a \cdot \frac{1}{\partial^+} W_{ma} \right\}_| + c.c., \quad (\text{C.1})$$

where we used (2.43) and the fact that  $d_n W_{ma} = 0$ . For the  $M$ -independent part, we have

$$\mathcal{H}_{BLG}^{(1)} = -\frac{1}{48} f^{abcd} \varepsilon_{mnkl} \times \\ \times d^{[4]} \left\{ \frac{\partial}{\partial^+} q^m \bar{\phi}_a \cdot \partial^+ \bar{\phi}_b \cdot \frac{1}{\partial^+} (\partial^+ \bar{\phi}_c \cdot q^{nkl} \bar{\phi}_d + 3\partial^+ q^n \bar{\phi}_c \cdot q^{kl} \bar{\phi}_d) \right\}_| + c.c. \quad (\text{C.2})$$

For the “ $C$ -only” part of the projection, only the term with “3” contributes, and there is only one way in which the four derivatives in  $d^{[4]} = \frac{1}{4!} \varepsilon_{rstu} d^{rstu}$  should be distributed among the four  $\bar{\phi}$ 's. We thus immediately find that <sup>8</sup>

$$\mathcal{H}_{BLG|C^4}^{(1)} = \frac{1}{2} \frac{\partial}{\partial^+} C^{mn} \cdot \partial^+ C_{mi} \cdot \frac{1}{\partial^+} (\partial^+ C^{ij} \cdot C_{nj}) + c.c. \quad (\text{C.3})$$

This corresponds to equation (H.10) in [18], which there took much more effort to derive. Our simplified derivation is the consequence of using the expression of  $W_{ma}$  in terms of  $q$ 's (rather than in terms of  $d$ 's). Using the identity (B.5) and the antisymmetry of  $f^{abcd}$ , we find

$$\mathcal{H}_{BLG|C^4}^{(1)} = \frac{1}{4} \left\{ \frac{\partial}{\partial^+} C_1 \cdot \partial^+ C_2 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_1 - \partial^+ C_1 \cdot C_2) + \frac{\partial}{\partial^+} C_1 \cdot \partial^+ C_1 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_2) \right\} \\ = \frac{1}{4} \frac{\partial}{\partial^+} C_1 \cdot \partial^+ \left[ C_1 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_2) \right] = -\frac{1}{4} \partial C_1 \cdot C_1 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_2), \quad (\text{C.4})$$

which agrees with (3.7). We verified that  $\mathcal{H}_{BLG|A^2C^2}^{(1)}$  and  $\mathcal{H}_{BLG|A^4}^{(1)}$  match with (3.7) as well, together with  $\mathcal{H}_{BLG|C^4}^{(1)}$  giving

$$\mathcal{H}_{BLG|X}^{(1)} = -(A \cdot \partial \bar{A} + \bar{A} \cdot \partial A + \frac{1}{2} C_1 \cdot \partial C_1) \cdot \frac{1}{\partial^+} (A \cdot \partial^+ \bar{A} + \bar{A} \cdot \partial^+ A + \frac{1}{2} C_2 \cdot \partial^+ C_2). \quad (\text{C.5})$$

Turning to the  $M$ -dependent part of  $\mathcal{H}^{(1)}$ , we find that

$$\mathcal{H}_M^{(1)} = -\frac{M}{48\sqrt{2}} f^{abcd} \varepsilon_{mnkl} \times \\ \times d^{[4]} \left\{ \frac{1}{\partial^{+2}} V^m \bar{\phi}_a \cdot \partial^+ \bar{\phi}_b \cdot \frac{1}{\partial^+} (\partial^+ \bar{\phi}_c \cdot q^{nkl} \bar{\phi}_d + 3\partial^+ q^n \bar{\phi}_c \cdot q^{kl} \bar{\phi}_d) \right\}_| + c.c., \quad (\text{C.6})$$

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<sup>8</sup> It is convenient to keep the gauge indices, together with  $f^{abcd}$ , implicit. The antisymmetry of  $f^{abcd}$  translates into the fermionic-like behavior of the four objects separated by the central dots.

where

$$V^m \equiv Z\Omega_n^m q^n + \Omega_n^m q_k q^{nk} + \Omega_n^k q_k q^{mn} , \quad (\text{C.7})$$

which has the following projections

$$d^r V^m \bar{\phi}_| = -Z\Omega_p^r d^{pm} \bar{\phi}_| , \quad \varepsilon_{rstu} d^{rst} V^m \bar{\phi}_| = -12Z\Omega_u^m \partial^{+2} \phi_|. \quad (\text{C.8})$$

Acting with  $d^{[4]} = \frac{1}{4!} \varepsilon_{rstu} d^{rstu}$ , we easily find that the “ $A$ -only” part of  $\mathcal{H}_M^{(1)}$  vanishes,

$$\boxed{\mathcal{H}_{M|A^4}^{(1)} = 0} , \quad (\text{C.9})$$

as it is proportional to  $\Omega_m^m = 0$ . For the part with two  $A$ ’s and two  $C$ ’s, we find

$$\begin{aligned} \mathcal{H}_{M|A^2 C^2}^{(1)} = & \frac{i}{2} M \Omega_n^m \left\{ \frac{1}{\partial^+} C^{nk} \cdot \partial^+ C_{mk} \cdot \frac{1}{\partial^+} (\bar{A} \cdot \partial^+ A) + A \cdot \bar{A} \cdot \frac{1}{\partial^+} (\partial^+ C^{nk} \cdot C_{mk}) \right. \\ & \left. - \frac{1}{\partial^+} C^{nk} \cdot \bar{A} \cdot \frac{1}{\partial^+} (\partial^+ C_{mk} \cdot \partial^+ A) + \frac{1}{\partial^+} C^{nk} \cdot \bar{A} \cdot \frac{1}{\partial^+} (\partial^{+2} A \cdot C_{mk}) \right\} + c.c. \end{aligned} \quad (\text{C.10})$$

Using the total antisymmetry of  $f^{abcd}$ , complex conjugation rules

$$(A)^* = \bar{A}, \quad (C_{mn})^* = C^{mn}, \quad (\Omega_m^n)^* = -\Omega_n^m , \quad (\text{C.11})$$

and the following identity

$$\Omega_n^m (C^{mi}, C_{mi}) = -\Omega_n^m (C_{mi}, C^{mi}) , \quad (\text{C.12})$$

which follows from (B.2) and  $\Omega_m^m = 0$ , it is straightforward to prove that

$$\boxed{\mathcal{H}_{M|A^2 C^2}^{(1)} = i M \Omega_n^m (C^{nk} \cdot C_{mk} \cdot A \cdot \bar{A})} . \quad (\text{C.13})$$

For the “ $C$ -only” part, we find

$$\mathcal{H}_{M|C^4}^{(1)} = -\frac{i}{2} M \Omega_k^m \left( \frac{1}{\partial^+} C^{kn} \cdot \partial^+ C_{mi} \cdot \frac{1}{\partial^+} (\partial^+ C^{ij} \cdot C_{nj}) \right) + c.c. \quad (\text{C.14})$$

In order to simplify this, it helps to split  $i\Omega_k^m C^{kn}$  into two parts, symmetric in  $mn$  and anti-symmetric in  $mn$ ,

$$S^{mn} = \frac{1}{2} (V^{mn} + V^{nm}), \quad A^{mn} = \frac{1}{2} (V^{mn} - V^{nm}); \quad V^{mn} \equiv i\Omega_k^m C^{kn} . \quad (\text{C.15})$$

Note that  $(A^{mn})^* = -\frac{1}{2} \varepsilon_{mnkl} A^{kl}$ . Using (B.2), we find that

$$\frac{1}{\partial^+} A^{mn} \cdot \partial^+ C_{mi} \cdot \frac{1}{\partial^+} (\partial^+ C^{ij} \cdot C_{nj}) = -\frac{1}{4} A^{mn} \cdot C_{mn} \cdot \frac{1}{\partial^+} (\partial^+ C_{ij} \cdot C^{ij}) . \quad (\text{C.16})$$

As this expression is purely imaginary,  $A^{mn}$  does not contribute to  $\mathcal{H}_{M|C^4}^{(1)}$ . Turning to the contribution of  $S^{mn}$ , we note that (B.2) and  $S^{mn} C_{mn} = 0$  imply that  $(S^{mn}, C_{mi}, C^{ij}, C_{nj})$  is totally antisymmetric in the last three arguments. It is then straightforward to show that

$$\frac{1}{\partial^+} S^{mn} \cdot \partial^+ C_{mi} \cdot \frac{1}{\partial^+} (\partial^+ C^{ij} \cdot C_{nj}) = \frac{1}{6} (S^{mn} \cdot C^{ij} \cdot C_{mi} \cdot C_{nj}) , \quad (\text{C.17})$$

which is real. As a result,

$$\boxed{\mathcal{H}_{M|C^4}^{(1)} = -\frac{i}{6} M \Omega_k^m (C^{kn} \cdot C^{ij} \cdot C_{mi} \cdot C_{nj})} . \quad (\text{C.18})$$

Finally, using that  $\Omega_m^n = (\eta_3)_{mn}$  and the expressions (2.12), we find

$$\begin{aligned} \Omega_n^m (C^{mk} \cdot C_{mk} \cdot A \cdot \bar{A}) &= \Omega_n^m (C_{mk} \cdot C^{kn} \cdot A \cdot \bar{A}) \\ &= 2! \text{Tr}(\eta_3 \eta_1 \eta_2) \frac{1}{2} (X^5 \cdot X^6 \cdot A \cdot \bar{A}) \\ &= 4(X^5 \cdot X^6 \cdot A \cdot \bar{A}) \\ &= -4i(X^5 \cdot X^6 \cdot X^3 \cdot X^4) , \\ \Omega_k^m (C^{kn} \cdot C^{ij} \cdot C_{mi} \cdot C_{nj}) &= \Omega_k^m (C_{mi} \cdot C^{ij} \cdot C_{jn} \cdot C^{nk}) \\ &= 4! \text{Tr}(\eta_3 \eta_3 \tilde{\eta}_1 \tilde{\eta}_2 \tilde{\eta}_3) \frac{1}{4} (X^7 \cdot (-iX^8) \cdot (iX^9) \cdot (-iX^{10})) \\ &= 4!i(X^7 \cdot X^8 \cdot X^9 \cdot X^{10}) , \end{aligned} \quad (\text{C.19})$$

so that the sum of (C.9), (C.13) and (C.18) gives

$$\begin{aligned} \mathcal{H}_{M|X}^{(1)} &= iM \Omega_n^m (C^{mk} \cdot C_{mk} \cdot A \cdot \bar{A}) - \frac{i}{6} M \Omega_k^m (C^{kn} \cdot C^{ij} \cdot C_{mi} \cdot C_{nj}) \\ &= 4M(X^3 \cdot X^4 \cdot X^5 \cdot X^6 + X^7 \cdot X^8 \cdot X^9 \cdot X^{10}) , \end{aligned} \quad (\text{C.20})$$

in agreement with (3.7).

## D The quadratic in $f^{abcd}$ part of the Hamiltonian

The part of the quadratic form Hamiltonian (3.10) quadratic in  $f^{abcd}$  is

$$\mathcal{H}^{(2)} = \frac{i}{16\sqrt{2}} \int d^4\theta d^4\bar{\theta} W_{ma} \frac{1}{\partial^+} W_a^m . \quad (\text{D.1})$$

More explicitly,

$$\begin{aligned} \mathcal{H}^{(2)} &= \frac{i}{16\sqrt{2} \cdot 18} \varepsilon_{mnpq} \varepsilon^{mrst} f^{abcd} f^{ab'c'd'} d^{[4]} d_{[4]} \\ &\times \left\{ \partial^+ \bar{\phi}_b \cdot \frac{1}{\partial^+} (\partial^+ \bar{\phi}_c \cdot q^{npq} \bar{\phi}_d + 3\partial^+ q^n \bar{\phi}_c \cdot q^{pq} \bar{\phi}_d) \right. \\ &\quad \left. \cdot \frac{1}{\partial^{+3}} (\partial^+ \phi_{b'} \cdot \frac{1}{\partial^+} (\partial^+ \phi_{c'} \cdot q_{rst} \phi_{d'} + 3\partial^+ q_r \phi_{c'} \cdot q_{st} \phi_{d'})) \right\} . \end{aligned} \quad (\text{D.2})$$

Thanks to our choice of writing  $W$ 's in terms of  $q$ 's,  $d_{[4]} = \frac{1}{4!} \varepsilon^{ijkl} d_{ijkl}$  goes through the second line (since  $d_m \bar{\phi} = 0$ ). Acting on the third line,  $d_{[4]}$  yields 30 different terms. Then we have to act with  $d^{[4]} = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} d^{\alpha\beta\gamma\delta}$ , with each derivative capable of hitting each of the six  $\phi$ 's in each of the 60 terms. Some of the resulting terms vanish as  $d^m \phi = 0$ , but still many remain. Instead of writing them all at once, it helps to organize the terms by their field content. Concentrating on the “ $A$ -only” part of  $\mathcal{H}^{(2)}$ , we collect terms with 0 or 4  $d$ 's (or  $q$ 's, which become  $d$ 's upon

projection) remaining on each  $\phi$  after  $\{d^m, d_n\} = -Z\delta_n^m$  is used. We find

$$\begin{aligned}
\mathcal{H}_{|A^6}^{(2)} &= \frac{i}{16\sqrt{2} \cdot 18} \varepsilon_{mnpq} \varepsilon^{mrst} \frac{1}{4!} \varepsilon^{ijkl} \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} (-4) \partial^+ \bar{\phi} \cdot \frac{1}{\partial^+} (\partial^+ \bar{\phi} \cdot d^\alpha q^{npq} \bar{\phi}) \\
&\cdot \frac{1}{\partial^+} \left[ 4\partial^+ d^{\beta\gamma\delta} d_{ijk} \phi \cdot \frac{1}{\partial^+} (\partial^+ \phi \cdot d_l q_{rst} \phi) \right. \\
&\quad + 3 \times 6\partial^+ d^{\beta\gamma} d_{ij} \phi \cdot \frac{1}{\partial^+} (2\partial^+ d^\delta d_k \phi \cdot d_l q_{rst} \phi + \partial^+ \phi \cdot d^\delta d_{kl} q_{rst} \phi) \\
&\quad + 3 \times 4\partial^+ d^\beta d_i \phi \cdot \frac{1}{\partial^+} (3\partial^+ d^{\gamma\delta} d_{jk} \phi \cdot d_l q_{rst} \phi - 2 \times 3\partial^+ d^\gamma d_j \phi \cdot d^\delta d_{kl} q_{rst} \phi \\
&\quad \quad \quad + \partial^+ \phi \cdot d^{\gamma\delta} d_{jkl} q_{rst} \phi) \\
&\quad \left. + \partial^+ \phi \cdot \frac{1}{\partial^+} (4\partial^+ d^{\beta\gamma\delta} d_{ijk} \phi \cdot d_l q_{rst} \phi + 3 \times 6\partial^+ d^{\beta\gamma} d_{ij} \phi \cdot d^\delta d_{kl} q_{rst} \phi \right. \\
&\quad \quad \left. + 3 \times 4\partial^+ d^\beta d_i \phi \cdot d^{\gamma\delta} d_{jkl} q_{rst} \phi + \partial^+ \phi \cdot d^{\beta\gamma\delta} d_{ijkl} q_{rst} \phi) \right] , \quad (D.3)
\end{aligned}$$

where we omitted  $f^{abcd} f^{ab'c'd'}$  while keeping the order of  $\phi$ 's fixed. After a bit of algebra, we find that the 10 terms inside the square bracket combine into the  $\partial^+{}^3$  derivative of a single term. Rewriting the result in terms of  $A$ 's, we obtain

$$\mathcal{H}_{|A^6}^{(2)} = 4f^{abcd} f^{ab'c'd'} \bar{A}_b \cdot \frac{1}{\partial^+} (\bar{A}_c \cdot \partial^+ A_d) \cdot A_{b'} \cdot \frac{1}{\partial^+} (A_{c'} \cdot \partial^+ \bar{A}_{d'}) . \quad (D.4)$$

Comparing this with the corresponding part in (3.7), we find that the two expressions agree.<sup>9</sup> Turning to the “ $C$ -only” part of (D.2), we find that

$$\begin{aligned}
\mathcal{H}_{|C^6}^{(2)} &= \frac{i}{16\sqrt{2} \cdot 18} \varepsilon_{mnpq} \varepsilon^{mrst} \frac{1}{4!} \varepsilon^{ijkl} \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\delta} (-2 \times 6) \partial^+ d^{\alpha\beta} \bar{\phi} \cdot \frac{1}{\partial^+} (3\partial^+ d^\gamma q^n \bar{\phi} \cdot q^{pq} \bar{\phi}) \\
&\cdot \frac{1}{\partial^+} \left[ 4\partial^+ d^\delta d_{ijk} \phi \cdot \frac{1}{\partial^+} (3\partial^+ d_l q_r \phi \cdot q_{st} \phi) \right. \\
&\quad \left. + 6\partial^+ d_{ij} \phi \cdot \frac{1}{\partial^+} (3\partial^+ d^\delta d_{kl} q_r \phi \cdot q_{st} \phi - 6\partial^+ d_k q_r \phi \cdot d^\delta d_l q_{st} \phi) \right] . \quad (D.5)
\end{aligned}$$

The three terms inside the square bracket combine into the  $\partial^+$  derivative of a single term, and rewriting the result in terms of  $C$ 's, we obtain

$$\mathcal{H}_{|C^6}^{(2)} = \partial^+ C_{ij} \cdot \frac{1}{\partial^+} (\partial^+ C^{mi} \cdot C_{mn}) \cdot \frac{1}{\partial^+} \left[ \partial^+ C^{jk} \cdot \frac{1}{\partial^+} (\partial^+ C_{kl} \cdot C^{lm}) \right] . \quad (D.6)$$

The identity (B.8) allows to rewrite this as

$$\begin{aligned}
\mathcal{H}_{|C^6}^{(2)} &= -\frac{1}{16} \left\{ \partial^+ C_3 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_2) \cdot \frac{1}{\partial^+} \left[ \partial^+ C_3 \cdot \frac{1}{\partial^+} (\partial^+ C_1 \cdot C_1) + \partial^+ C_1 \cdot C_1 \cdot C_3 \right] \right. \\
&\quad + \partial^+ C_3 \cdot C_1 \cdot C_2 \cdot \frac{1}{\partial^+} \left[ \partial^+ C_3 \cdot \frac{1}{\partial^+} (\partial^+ C_1 \cdot C_2) + \partial^+ C_1 \cdot C_2 \cdot C_3 \right] \\
&\quad \left. + \partial^+ C_1 \cdot C_1 \cdot C_2 \cdot \frac{1}{\partial^+} \left[ \partial^+ C_2 \cdot \frac{1}{\partial^+} (\partial^+ C_3 \cdot C_3) + \partial^+ C_3 \cdot C_3 \cdot C_2 \right] \right\} . \quad (D.7)
\end{aligned}$$

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<sup>9</sup> The “ $A$ -only” part of the third line in (3.7) vanishes identically. Therefore, it would be incorrect to argue that the matching of (D.2) with (3.7) for the “ $A$ -only” part, together with the  $SO(8)$   $R$ -symmetry of  $\mathcal{H}^{(2)}$ , guarantees that they match for other parts as well.

Each square bracket can be written as a total  $\partial^+$  derivative of a single term. The sum of the 1st and 3rd lines similarly yields a total  $\partial^+$  derivative on the first triplet of  $C$ 's. Finally, using the following identity

$$\partial^+ C_3 \cdot C_1 \cdot C_2 \cdot \frac{1}{\partial^+} \left[ C_3 \cdot \frac{1}{\partial^+} (\partial^+ C_1 \cdot C_2) \right] = -\frac{1}{6} C_1 \cdot C_2 \cdot C_3 \cdot C_1 \cdot C_2 \cdot C_3 , \quad (\text{D.8})$$

we find that

$$\mathcal{H}_{|C^6}^{(2)} = \frac{1}{16} C_1 \cdot \frac{1}{\partial^+} (\partial^+ C_2 \cdot C_2) \cdot C_1 \cdot \frac{1}{\partial^+} (\partial^+ C_3 \cdot C_3) + \frac{1}{96} C_1 \cdot C_2 \cdot C_3 \cdot C_1 \cdot C_2 \cdot C_3 , \quad (\text{D.9})$$

which matches the corresponding part in (3.7). We also verified that the  $\mathcal{H}_{|A^2 C^4}^{(2)}$  and  $\mathcal{H}_{|A^4 C^2}^{(2)}$  parts of (D.2) and (3.7) match as well, giving

$$\begin{aligned} \mathcal{H}_{|A^2 C^4}^{(2)} &= \frac{1}{4} A \cdot \frac{1}{\partial^+} (C_1 \cdot \partial^+ C_1) \cdot \bar{A} \cdot \frac{1}{\partial^+} (C_2 \cdot \partial^+ C_2) + \frac{1}{8} A \cdot C_1 \cdot C_2 \cdot \bar{A} \cdot C_1 \cdot C_2 \\ &\quad + \frac{1}{4} C_1 \cdot \frac{1}{\partial^+} (C_2 \cdot \partial^+ C_2) \cdot C_1 \cdot \frac{1}{\partial^+} (A \cdot \partial^+ \bar{A} + c.c.) , \\ \mathcal{H}_{|A^4 C^2}^{(2)} &= A \cdot \frac{1}{\partial^+} (A \cdot \partial^+ \bar{A}) \cdot \bar{A} \cdot \frac{1}{\partial^+} (C_1 \cdot \partial^+ C_1) + c.c. \\ &\quad + C_1 \cdot \frac{1}{\partial^+} (\bar{A} \cdot \partial^+ A) \cdot C_1 \cdot \frac{1}{\partial^+} (A \cdot \partial^+ \bar{A}) . \end{aligned} \quad (\text{D.10})$$

In the  $\mathcal{H}_{|A^2 C^4}^{(2)}$  case, we had to use the identity (B.5). In all the cases, we used the  $[bcd]$  antisymmetry of  $f^{abcd}$  and performed various integrations by parts. However, surprisingly, the Fundamental Identity (3.2) was never needed in this analysis.

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